KMA315 Analysis 3A: Problems 3

Solutions to these problems should be submitted by 2:00pm on Tuesday the 12[th] *of April* 2016.

1. Let:

(i) $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions;

(ii) $S = \{x \in \mathbb{R} : f(x) \geq g(x)\}$; and

(iii) $(x_n)_{n=0}^{\infty}$ be a sequence of points from S.

Show that if $\lim_{n\to\infty} x_n$ exists then $\lim_{n\to\infty} x_n \in S$. *(5 marks)*

2. Let $f : [0, 1] \to [0, 1]$ be the function defined by

$$f(x) = \begin{cases} x & \text{when } x \in \mathbb{Q}; \text{ and} \\ 1 - x & \text{when } x \in \mathcal{C}(\mathbb{Q}). \end{cases}$$

Prove that:

(i) f assumes every value between 0 and 1 (ie. that f is surjective); *(1 mark)*

(ii) f is continuous only at $x = \frac{1}{2}$. *(2 marks)*

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Establish what value $f(x)$ takes for irrational values of x. *(3 marks)*

There are more questions over the page...

Nicholas Ham, Room 425

4. Let $(f_n)_{n=0}^{\infty}$ be the sequence of real-valued functions on $\mathbb{R}$ where for each $n \in \mathbb{N}$,

$$f_n(x) = x + \frac{1}{n} \text{ for all } x \in \mathbb{R}.$$

Establish that:

(i) $(f_n)_{n=0}^{\infty}$ converges uniformly on $\mathbb{R}$; *(2 marks)*

(ii) $(f_n^2)_{n=0}^{\infty}$ does not converge uniformly on $\mathbb{R}$. *(3 marks)*
Note: for each $n \in \mathbb{N}$, $f_n^2(x) = [f_n(x)]^2$ for all $x \in \mathbb{R}$.

5. Let $(f_n)_{n=0}^{\infty}$ be the sequence of real-valued functions on $[0, 1]$ where for each $n \in \mathbb{N}$,

$$f_n(x) = x^n \text{ for all } x \in [0, 1].$$

(i) Establish whether $(f_n)_{n=0}^{\infty}$ converges pointwise; *(1 mark)*

(ii) if it does, find the pointwise limit of $(f_n)_{n=0}^{\infty}$. *(1 mark)*

Nicholas Ham, Room 425